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Asymptotic formulae for two-variable Hermite polynomials

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Abstract. New asymptotic formulae for the two-dimensional Hermite polynomials with large values of indices are found. The applications to the photon distribution functions in squeezed one-mode mixed quantum states are considered.

1. Introduction

In 1864, Hermite [1] introduced a class of polynomials of both one and several variables, whose generating functions were exponentials of some quadratic forms. The numerous applications of the one-dimensional Hermite polynomials are well known: it is sufficient to mention the solution of the harmonic-oscillator problem in quantum mechanics. As to the Hermite polynomials of several variables, they find useful applications, e.g., in the kinetic theory of gases [2] and in the theories of fluctuations [3], optical systems [4, 5], multidimensional quantum systems with quadratic time-dependent Hamiltonians [6] (see also [7]) and quantum squeezed states [8–11]. However, in neither the papers mentioned nor in the most comprehensive books on special functions [12, 13] can one find any *asymptotic formula* for multidimensional Hermite polynomials.

The aim of the present paper is to partly remove this drawback from the theory of special functions, i.e. to obtain asymptotic expressions for *two-dimensional* Hermite polynomials. In the special case of two-dimensional polynomials *of zero arguments* this problem was solved recently in [14] by means of reducing these polynomials to the classical (one-dimensional) orthogonal polynomials (the Gegenbauer polynomials) the asymptotics of which are well known. The approach used in the present paper, which is based on the remarkable integral representation for the multidimensional Hermite polynomials found by Feldheim [15], enables one to consider the general case of *non-zero arguments*.

The formulae derived are applied to the physical problem of photon statistics in generic one-mode squeezed mixed quantum states.

2. Integral representations for the two-dimensional Hermite polynomials

We use the definition of the two-dimensional Hermite polynomials $H_{mn}^{\{\mathbf{R}\}}(y_1, y_2)$ [13]

$$\exp[-\tfrac{1}{2}\boldsymbol{a}\mathbf{R}\boldsymbol{a} + \boldsymbol{a}\mathbf{R}\boldsymbol{y}] = \sum_{n,m=0}^{\infty} \frac{a_1^m a_2^n}{m!n!} H_{mn}^{\{\mathbf{R}\}}(\boldsymbol{y}). \tag{2.1}$$

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Here, a_1 and a_2 are arbitrary complex numbers combined into a vector $\boldsymbol{a} = (a_1, a_2)$

$$\boldsymbol{a}\mathbf{R}\boldsymbol{a} = \sum_{i,k=0}^{2} a_i R_{ik} a_k \qquad \boldsymbol{a}\mathbf{R}\boldsymbol{y} = \sum_{i,k=0}^{2} a_i R_{ik} y_k \qquad \boldsymbol{y} = (y_1, y_2)$$

and $\mathbf{R}$ is a symmetric matrix

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{pmatrix}.$$

Another definition was used by Feldheim [15]. He considered the following polynomials $F_{mn}^{(\lambda,\mu)}(\boldsymbol{x})$ defined as

$$\exp[-\boldsymbol{h}\mathbf{F}\boldsymbol{h} + 2\boldsymbol{h}\mathbf{F}\boldsymbol{x}] = \sum_{n,m=0}^{\infty} \frac{h_1^m h_2^n}{m!n!} F_{mn}^{(\lambda,\mu)}(\boldsymbol{x}) \tag{2.2}$$

where matrix $\mathbf{F}$ reads

$$\mathbf{F} = \begin{pmatrix} \lambda^2 - 1 & -1 \\ -1 & \mu^2 - 1 \end{pmatrix}.$$

Comparing equations (2.1) and (2.2) and introducing the notation

$$\rho = -R_{12} \qquad r_{11} = R_{11} + \rho \qquad r_{22} = R_{22} + \rho \tag{2.3}$$

we may write the following relation between two types of polynomials:

$$H_{mn}^{\{\mathbf{R}\}}(y_1, y_2) = (\rho/2)^{(m+n)/2} F_{mn}^{(\lambda,\mu)}\left(y_1\sqrt{\rho/2}, y_2\sqrt{\rho/2}\right) \tag{2.4}$$

$$\lambda^2 = r_{11}/\rho \qquad \mu^2 = r_{22}/\rho. \tag{2.5}$$

Imposing on the parameters λ and μ the restrictions

$$|\lambda| > 1 \qquad |\mu| > 1 \qquad \lambda^{-2} + \mu^{-2} < 1 \tag{2.6}$$

Feldheim obtained in [15] (having made the reference to his preceding papers [16]) the following integral representation:

$$F_{mn}^{(\lambda,\mu)}(x_1, x_2) = \frac{1}{\sqrt{\pi}} \lambda^m \mu^n \int_{-\infty}^{\infty} e^{-z^2} H_m\left(\frac{z+\xi}{\lambda}\right) H_n\left(\frac{z+\eta}{\mu}\right) dz$$

where

$$\xi = (\lambda^2 - 1)x_1 - x_2 \qquad \eta = -x_1 + (\mu^2 - 1)x_2$$

and $H_m(z)$ is the usual Hermite polynomial [13] defined as

$$\exp[-h^2 + 2hz] = \sum_{m=0}^{\infty} \frac{h^m}{m!} H_m(z).$$

Due to equation (2.4), we get the following integral representation for the two-dimensional Hermite polynomials:

$$H_{mn}^{\{\mathbf{R}\}}(y_1, y_2) = \frac{1}{\sqrt{\pi}}\left(\frac{r_{11}}{2}\right)^{m/2}\left(\frac{r_{22}}{2}\right)^{n/2}\int_{-\infty}^{\infty} \mathrm{e}^{-z^2} H_m\left(\frac{\sqrt{2\rho}z + Y_1}{\sqrt{2r_{11}}}\right) H_n\left(\frac{\sqrt{2\rho}z + Y_2}{\sqrt{2r_{22}}}\right) \mathrm{d}z \tag{2.7}$$

where

$$Y_1 = R_{11}y_1 - \rho y_2 \qquad Y_2 = R_{22}y_2 - \rho y_1.$$

Restrictions (2.6) can be rewritten as

$$\left|\frac{r_{11}}{\rho}\right| > 1 \qquad \left|\frac{r_{22}}{\rho}\right| > 1 \qquad \frac{\rho}{r_{11}} + \frac{\rho}{r_{22}} < 1. \tag{2.8}$$

The last of these conditions is equivalent to the inequality (provided $r_{11}r_{22} > 0$)

$$\Delta \equiv \det \mathbf{R} \equiv R_{11}R_{22} - \rho^2 > 0. \tag{2.9}$$

Replacing each one-dimensional Hermite polynomial on the right-hand side of equation (2.7) with its integral representation [13]

$$H_n(x) = \frac{2^{n+1}}{\sqrt{\pi}}\mathrm{e}^{x^2}\int_0^{\infty} \mathrm{e}^{-t^2} t^n \cos\left(2xt - n\frac{\pi}{2}\right) \mathrm{d}t$$

we obtain a *Gaussian* integral with respect to z, which can be easily calculated. In this way, we arrive at the following expression:

$$\begin{aligned} H_{mn}^{\{\mathbf{R}\}}(y_1, y_2) = {} & \frac{1}{\pi\sqrt{\Delta}}(2r_{11})^{(m+1)/2}(2r_{22})^{(n+1)/2}\int_0^{\infty}\mathrm{d}t_1\int_0^{\infty}\mathrm{d}t_2\, t_1^m t_2^n \\ & \times \exp\left[\frac{1}{2}(R_{11}y_1^2 + R_{22}y_2^2 - 2\rho y_1 y_2) - \frac{1}{\Delta}(r_{11}R_{22}t_1^2 + r_{22}R_{11}t_2^2)\right] \\ & \times \left\{\exp\left(-\frac{2\rho}{\Delta}\sqrt{r_{11}r_{22}}t_1t_2\right)\cos\left[\sqrt{2r_{11}}y_1t_1 + \sqrt{2r_{22}}y_2t_2 - \frac{m+n}{2}\pi\right]\right. \\ & \left. + \exp\left(\frac{2\rho}{\Delta}\sqrt{r_{11}r_{22}}t_1t_2\right)\cos\left[\sqrt{2r_{11}}y_1t_1 - \sqrt{2r_{22}}y_2t_2 - \frac{m-n}{2}\pi\right]\right\}. \end{aligned} \tag{2.10}$$

Thus, we have to evaluate the integrals with the following structure:

$$I = \int_0^{\infty}\mathrm{d}t_1\int_0^{\infty}\mathrm{d}t_2\,\exp[-at_1^2 - bt_2^2 - ct_1t_2 + m\ln t_1 + n\ln t_2 + \mathrm{i}ft_1 + \mathrm{i}gt_2] \tag{2.11}$$

where

$$a = \frac{r_{11}R_{22}}{\Delta} \qquad b = \frac{r_{22}R_{11}}{\Delta} \qquad c = \pm\frac{2\rho}{\Delta}\sqrt{r_{11}r_{22}} \tag{2.12}$$

$$f = \pm\sqrt{2r_{11}}y_1 \qquad g = \pm\sqrt{2r_{22}}y_2. \tag{2.13}$$

For large values of integers m and n this can be achieved by means of the steepest-descent method. The saddle point is determined from the equations (see appendix for details)

$$-2at_1 - ct_2 + \frac{m}{t_1} + \mathrm{i}f = 0 \tag{2.14}$$

$$-2bt_2 - ct_1 + \frac{n}{t_2} + \mathrm{i}g = 0. \tag{2.15}$$

Designating the solutions to these equations by τ_1 and τ_2 and taking into account their consequence

$$a\tau_1^2 + b\tau_2^2 + c\tau_1\tau_2 = \tfrac{1}{2}(m + n + \mathrm{i}f\tau_1 + \mathrm{i}g\tau_2)$$

we get the following asymptotics for $m, n \gg 1$ (see equation (A.4)):

$$I \approx 2\pi \left[4ab - c^2 + \frac{mn}{(\tau_1\tau_2)^2} + \frac{2an}{\tau_2^2} + \frac{2bm}{\tau_1^2}\right]^{-1/2} \tau_1^m \tau_2^n \exp[\tfrac{1}{2}(\mathrm{i}f\tau_1 + \mathrm{i}g\tau_2 - m - n)]. \tag{2.16}$$

3. The case of zero arguments

Equations (2.14) and (2.15) can be easily solved in the case of $f = g = 0$, which corresponds to polynomials with zero arguments. Consider first the 'diagonal' polynomials with $m = n$. Then one gets immediately the relation $a\tau_1^2 = b\tau_2^2$. Since we integrate in equation (2.11) from 0 to $+\infty$ both over τ_1 and τ_2, we must choose (bearing in mind that the real parts of coefficients a and b must be positive to guarantee the convergence of the integral (2.11)) $\tau_2 = \tau_1\sqrt{a/b}$. Then we have the following solutions:

$$\tau_1 = \left(\frac{b}{a}\right)^{1/4} \left(\frac{m}{2\sqrt{ab} + c}\right)^{1/2} \qquad \tau_2 = \left(\frac{a}{b}\right)^{1/4} \left(\frac{m}{2\sqrt{ab} + c}\right)^{1/2}. \tag{3.1}$$

Putting them into equation (2.16), we get

$$I \approx \frac{\pi}{\sqrt{2}}(ab)^{-1/4} \left(\frac{m}{e}\right)^m (2\sqrt{ab} + c)^{-m-1/2}. \tag{3.2}$$

This formula, together with equations (2.10) and (2.12), leads to the following asymptotics of the 'diagonal' Hermite polynomials of zero arguments for $m \gg 1$:

$$H_{mm}^{\{\mathrm{R}\}}(0,0) \approx \left(\frac{mR}{e}\right)^m \{(1+\zeta)^{m+1/2} + (-1)^m(1-\zeta)^{m+1/2}\} \tag{3.3}$$

where

$$R = \sqrt{R_{11}R_{22}} \qquad \zeta = \rho/R. \tag{3.4}$$

It coincides with the formula obtained in [14] in the framework of a more complicated approach.

If $m \neq n$, we may exclude, e.g., variable τ_2 from equations (2.14) and (2.15). Then we arrive at the *biquadratic* equation for τ_1. Choosing the solution of this equation which coincides with (3.1) when $m = n$, we have

$$\tau_1 = \left[\left(\frac{2bm}{4ab-c^2}\right)\frac{\gamma-(n+m)c}{\gamma+(n-m)c}\right]^{1/2} \qquad \tau_2 = \left[\left(\frac{2an}{4ab-c^2}\right)\frac{\gamma-(n+m)c}{\gamma+(m-n)c}\right]^{1/2} \tag{3.5}$$

where

$$\gamma = [16abmn + c^2(m-n)^2]^{1/2}.$$

Thus, we obtain the following asymptotics of integral (2.11):

$$I \approx \frac{\pi}{\sqrt{\gamma}}\{\gamma-(m+n)c\}^{(m+n+1)/2}\left\{\frac{2m/e}{\gamma+c(n-m)}\right\}^{m/2}\left\{\frac{2n/e}{\gamma+c(m-n)}\right\}^{n/2}. \tag{3.6}$$

It results in the following generalization of equation (3.3):

$$\begin{aligned} H_{mn}^{(\mathbf{R})}(0,0) &\approx (mR_{11}/e)^{m/2}(nR_{22}/e)^{n/2} \\ &\quad\times\left\{\frac{[1+(m+n)z]^{(m+n+1)/2}}{[1+(m-n)z]^{m/2}[1+(n-m)z]^{n/2}}\cos\left(\frac{m-n}{2}\pi\right)\right. \\ &\quad\left.+\frac{[1-(m+n)z]^{(m+n+1)/2}}{[1+(n-m)z]^{m/2}[1+(m-n)z]^{n/2}}\cos\left(\frac{m+n}{2}\pi\right)\right\} \end{aligned} \tag{3.7}$$

where

$$z = \frac{\zeta}{\left[4mn+\zeta^2(m-n)^2\right]^{1/2}}. \tag{3.8}$$

Formula (3.7) can be simplified in two important special cases. The first corresponds to the 'quasi-diagonal' polynomials, when $(m-n)^2 \ll m+n$

$$\begin{aligned} H_{mn}^{(\mathbf{R})}(0,0) &\approx \left(\frac{m+n}{2e}\right)^{(m+n)/2} R_{11}^{m/2}R_{22}^{n/2} \\ &\quad\times\left\{\cos\left(\frac{m-n}{2}\pi\right)(1+\zeta)^{(m+n+1)/2}+\cos\left(\frac{m+n}{2}\pi\right)(1-\zeta)^{(m+n+1)/2}\right\}. \end{aligned} \tag{3.9}$$

The second case corresponds to the 'asymmetric' polynomials when $m \gg n^2$ and $m\rho^2 \gg nR_{11}R_{22}$. Then $mz \approx 1$ and the first term inside the figure brackets in equation (3.7) appears much greater than the second one. Thus, the leading term of the asymptotic expansion does not contain coefficient R_{22} at all

$$H_{mn}^{(\mathbf{R})}(0,0) \approx \sqrt{2}\cos\left(\frac{m-n}{2}\pi\right)\left(\frac{mR_{11}}{e}\right)^{m/2}\left(\frac{m}{R_{11}}\right)^{n/2}\rho^n. \tag{3.10}$$

In [14], the following asymptotics were found for $m-n \gg 1$ and $n \sim \mathcal{O}(1)$ (without any restriction on parameter R_{12}):

$$H_{mn}^{(\mathbf{R})}(0,0) \approx \cos\left(\frac{m+n}{2}\pi\right)\frac{\Gamma((m+1)/2)}{\sqrt{\pi}}(2R_{11})^{(m-n)/2}(-\Delta)^{n/2}H_n\left(R_{12}\sqrt{\frac{m-n}{-2\Delta}}\right) \tag{3.11}$$

with Δ as defined in equation (2.9). If $mR_{12}^2 \gg n^2\Delta$, then we may replace $H_n(x)$ by its leading term $(2x)^n$ and equation (3.11) turns into equation (3.10) due to Stirling's formula for the gamma function (remember that $R_{12} = -\rho$). This means that equation (3.10) holds, in fact, not only for large values of n (as was assumed in its derivation), but in the case of $n \sim \mathcal{O}(1)$ as well (provided the other restrictions are satisfied, of course).

4. Non-zero arguments

Now we proceed to the generic case when the arguments of the two-dimensional Hermite polynomial are not equal to zero. This means that $f \neq 0$ and $g \neq 0$ in equation (2.11). In such a case, the system of equations (2.14) and (2.15) is equivalent to a complete algebraic equation of the fourth order for t_1 or t_2, which cannot be solved in an explicit analytical form. Therefore, we assume that both parameters, f and g, are of the order of unity when $m \gg 1$ and $n \gg 1$. Then we may look for the approximate solutions of equations (2.14) and (2.15) in the form of the series

$$\tau_j = \tau_j^{(0)} + \tau_j^{(1)} + \tau_j^{(2)} + \cdots. \tag{4.1}$$

Putting this expansion into equations (2.14) and (2.15) yields the following equations for the first-order corrections:

$$\begin{aligned} &\{2a + m[\tau_1^{(0)}]^{-2}\}\tau_1^{(1)} + c\tau_2^{(1)} = \mathrm{i}f \\ &c\tau_1^{(1)} + \{2b + n[\tau_2^{(0)}]^{-2}\}\tau_2^{(1)} = \mathrm{i}g. \end{aligned} \tag{4.2}$$

The second-order corrections obey the similar equations

$$\begin{aligned} &\{2a + m[\tau_1^{(0)}]^{-2}\}\tau_1^{(2)} + c\tau_2^{(2)} = m[\tau_1^{(1)}]^2[\tau_1^{(0)}]^{-3} \\ &c\tau_1^{(2)} + \{2b + n[\tau_2^{(0)}]^{-2}\}\tau_2^{(2)} = n[\tau_2^{(1)}]^2[\tau_2^{(0)}]^{-3}. \end{aligned} \tag{4.3}$$

One can easily check that the kth term of the expansion (4.1) has the order of magnitude $\tau_j^{(k)} \sim \mathcal{O}(m^{(1-k)/2} f^k)$. Its contribution to the argument of the exponential in equation (A.4) has the order $\mathcal{O}(m^{(1-k)/2} f^{(k+1)})$. In particular, putting the expansion equation (4.1) into the right-hand side of equation (A.4) and taking into account equations (2.14), (2.15), (4.2) and (4.3), we arrive at the relation

$$I \approx I^{(0)} \exp\left[\mathrm{i}f\tau_1^{(0)} + \mathrm{i}g\tau_2^{(0)} + \frac{\mathrm{i}}{2}f\tau_1^{(1)} + \frac{\mathrm{i}}{2}g\tau_2^{(1)} + \mathcal{O}\left(\frac{f^3}{\sqrt{m}}\right)\right] \tag{4.4}$$

$I^{(0)}$ being given by equation (3.6). (Evidently, although we may replace τ_j with $\tau_j^{(0)}$ in the pre-exponential square-bracket term of equation (A.4), nonetheless, we must take into account all the corrections in the terms τ_1^m and τ_2^n.) Consequently, we have to calculate only the first-order corrections to the solutions (3.5) since $f^3/\sqrt{m} \ll 1$ for $f \sim \mathcal{O}(1)$. The explicit expressions for these corrections are as follows:

$$\tau_1^{(1)} = \mathrm{i}f\frac{[\gamma + (m-n)c](\gamma + nc)}{4a\gamma[\gamma + (m+n)c]} - \mathrm{i}g\frac{mnc}{\gamma[\gamma + (m+n)c]} \tag{4.5}$$

$$\tau_2^{(1)} = \mathrm{i}g\frac{[\gamma + (n-m)c](\gamma + mc)}{4b\gamma[\gamma + (m+n)c]} - \mathrm{i}f\frac{mnc}{\gamma[\gamma + (m+n)c]}. \tag{4.6}$$

Putting these expressions into equation (4.4) yields the formula

$$\begin{aligned} I \approx I^{(0)} \exp\Bigg\{ &\mathrm{i}f\left(\frac{m[\gamma + (m-n)c]}{2a[\gamma + (m+n)c]}\right)^{1/2} + \mathrm{i}g\left(\frac{n[\gamma + (n-m)c]}{2b[\gamma + (m+n)c]}\right)^{1/2} \\ &+ \frac{8abcmnfg - f^2 b[\gamma + (m-n)c](\gamma + nc) - g^2 a[\gamma + (n-m)c](\gamma + mc)}{8ab\gamma[\gamma + (m+n)c]}\Bigg\}. \end{aligned} \tag{4.7}$$

Calculating the integral on the right-hand side of equation (2.10) with the aid of this formula and using equation (2.13), we arrive after some algebra at the following asymptotic expression for the generic Hermite polynomial of two variables:

$$
\begin{aligned}
H_{mn}^{\{\mathrm{R}\}}(y_1, y_2) \approx{}& (mR_{11}/e)^{m/2}(nR_{22}/e)^{n/2}\exp[\tfrac{1}{2}\Phi(y_1, y_2)] \\
&\times\left\{u_+^{(m+n+1)/2}v_{mn}^{-m/2}v_{nm}^{-n/2}\cos\left(y_1\sqrt{\frac{m\Delta v_{nm}}{R_{22}u_-}} - y_2\sqrt{\frac{n\Delta v_{mn}}{R_{11}u_-}} - \frac{m-n}{2}\pi\right)\right. \\
&\times\exp\left(\frac{1}{4}\left[\rho u_+ y_1 y_2 - \frac{\Delta v_{nm}}{R_{22}u_-}(1-nz)y_1^2 - \frac{\Delta v_{mn}}{R_{11}u_-}(1-mz)y_2^2\right]\right) \\
&+ u_-^{(m+n+1)/2}v_{nm}^{-m/2}v_{mn}^{-n/2}\cos\left(y_1\sqrt{\frac{m\Delta v_{mn}}{R_{22}u_+}} + y_2\sqrt{\frac{n\Delta v_{nm}}{R_{11}u_+}} - \frac{m+n}{2}\pi\right) \\
&\left.\times\exp\left(\frac{1}{4}\left[\rho u_- y_1 y_2 - \frac{\Delta v_{mn}}{R_{22}u_+}(1+nz)y_1^2 - \frac{\Delta v_{nm}}{R_{11}u_+}(1+mz)y_2^2\right]\right)\right\}. \qquad (4.8)
\end{aligned}
$$

We have introduced the following notation:

$$
v_{mn} = 1 + (m-n)z \qquad u_\pm = 1 \pm (m+n)z \qquad \Phi = R_{11}y_1^2 + R_{22}y_2^2 - 2\rho y_1 y_2.
$$

Coefficients Δ and z were defined in equations (2.9) and (3.8). When $y_1 = y_2 = 0$, equation (4.8) turns into equation (3.7). For the 'diagonal' polynomial, we have a simpler expression

$$
\begin{aligned}
H_{mm}^{\{\mathrm{R}\}}(y_1, y_2) \approx{}& \left(\frac{mR}{e}\right)^m \exp(\tfrac{1}{4}[(1+2\zeta^2)(R_{11}y_1^2 + R_{22}y_2^2) - 3\rho y_1 y_2]) \\
&\times\left\{(1+\zeta)^{m+\frac{1}{2}}\cos\left[\sqrt{m(1+\zeta)}\left(\sqrt{R_{11}}y_1 - \sqrt{R_{22}}y_2\right)\right]\exp\left[-\frac{\zeta}{8}\Phi\right]\right. \\
&\left.+ (1-\zeta)^{m+\frac{1}{2}}\cos\left[\sqrt{m(1-\zeta)}\left(\sqrt{R_{11}}y_1 + \sqrt{R_{22}}y_2\right) - m\pi\right]\exp\left[\frac{\zeta}{8}\Phi\right]\right\}.
\end{aligned} \qquad (4.9)
$$

For the 'quasi-diagonal' polynomials (when $(m-n)^2 \ll (m+n)$), one should only insert the true phases $-(m-n)\pi/2$ and $-(m+n)\pi/2$ into the arguments of the cosine functions and change the amplitude factors in accordance with equation (3.9).

In the opposite limit case $m \gg n^2$, we have the following generalization of equation (3.10) (provided $m\zeta^2 \gg n$):

$$
H_{mn}^{\{\mathrm{R}\}}(y_1, y_2) \approx \sqrt{2}\left(\frac{mR_{11}}{e}\right)^{m/2}\left(\frac{m}{R_{11}}\right)^{n/2}\rho^n\cos\left(\sqrt{mR_{11}}\,y - \frac{m-n}{2}\pi\right)\exp\left(\tfrac{1}{4}R_{11}y^2\right) \qquad (4.10)
$$

where

$$
y = y_1 - \frac{\rho}{R_{11}}y_2.
$$

5. Applications: influence of quantum mixing on the photon distribution function in the squeezed states

Analysing all the steps which have led to the formulae obtained above, one may conclude that these formulae are valid provided both the coordinates of the saddle point given by equation (3.1) or (3.5) and the pre-exponential factor in equation (A.4) have positive real parts. For any real matrix $\mathbf{R}$, these conditions are equivalent to restrictions (2.8). It is interesting that these restrictions are fulfilled automatically when the two-dimensional Hermite polynomials are used to express the photon distribution function of the generic one-mode squeezed state (for reviews devoted to the squeezed states see, e.g., [17, 18]). The most remarkable feature of this distribution function, discovered independently in [9, 19–21], is its oscillating character for some sets of parameters. These oscillations manifest themselves in the most distinct way for the *pure* quantum states when the distribution function is reduced to the modulus squared of the usual (one-dimensional) Hermite polynomial (see, e.g., [22]). The influence of *quantum mixing* on the properties of squeezed states was investigated numerically in [9, 20, 23] where it was shown that mixing destroys the oscillations. Here, the same results will be obtained in an analytical form.

From the mathematical point of view, the so-called quadrature squeezed states may be considered as a certain subclass of the states described by means of the Gaussian density matrices (in the coordinate representation) or the Wigner functions (in the phase-space representation) [7, 24]. Any Gaussian distribution of two variables—the so-called quadrature components q and p—is determined completely by the first-order average values $\langle q\rangle$ and $\langle p\rangle$ and by their variances σ_{qq}, σ_{pp}, σ_{qp}. The photon distribution function corresponding to a generic Gaussian state is given by the 'diagonal' two-dimensional Hermite polynomial [11]

$$\mathcal{P}_n = \mathcal{P}_0 \frac{H_{nn}^{\{\mathbf{R}\}}(y_1, y_2)}{n!} \tag{5.1}$$

with the following elements of matrix $\mathbf{R}$

$$R_{11} = R_{22}^* = \frac{2(\sigma_{pp} - \sigma_{qq} - 2i\sigma_{pq})}{1 + 2T + 4d} \qquad R_{12} = \frac{1 - 4d}{1 + 2T + 4d} \tag{5.2}$$

and vector $\boldsymbol{y}$

$$y_1 = y_2^* = \frac{\sqrt{2}[(2\sigma_{pp} - 1 + 2\mathrm{i}\sigma_{pq})\langle q\rangle + \mathrm{i}(1 - 2\sigma_{qq} + 2\mathrm{i}\sigma_{pq})\langle p\rangle]}{2T - 4d - 1}. \tag{5.3}$$

Here

$$d = \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 \qquad T = \sigma_{pp} + \sigma_{qq} \tag{5.4}$$

are two independent invariants of the variance matrix

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{qq} & \sigma_{pq} \\ \sigma_{pq} & \sigma_{pp} \end{pmatrix}.$$

The probability of having no photons $\mathcal{P}_0$ is given by the formula

$$\mathcal{P}_0 = (d + \tfrac{1}{2}T + \tfrac{1}{4})^{-1/2} \exp\left[-\frac{\langle p\rangle^2(2\sigma_{qq} + 1) + \langle q\rangle^2(2\sigma_{pp} + 1) - 4\sigma_{pq}\langle p\rangle\langle q\rangle}{1 + 2T + 4d}\right]. \tag{5.5}$$

Due to the *generalized uncertainty relation* [7] (which ensures the positive definiteness of the statistical operator in the case under study), $d \geqslant \frac{1}{4}$. Consequently, $R_{12} \leqslant 0$. Since the photon distribution function is obviously invariant with respect to the rotations in the phase plane $p-q$, one may eliminate the covariance σ_{pq} by means of a suitable rotation of the coordinate axes in this plane. Moreover, one may always believe $\sigma_{pp} \geqslant \sigma_{qq}$ (redefining p and q, if necessary). Thus, without any loss of generality, we may assume that the coefficients R_{11}, R_{22} and $\rho = -R_{12}$ are non-negative. Then, the first two restrictions from (2.8) are fulfilled automatically. The most important is the third restriction $\det \mathbf{R} > 0$. Due to equation (5.2)

$$\det \mathbf{R} = \frac{2T - 4d - 1}{2T + 4d + 1}. \tag{5.6}$$

On the other hand, solving equations (5.4) with respect to σ_{pp} and σ_{qq}, we get the expressions (provided $\sigma_{pq} = 0$)

$$\sigma_{pp} = \tfrac{1}{2}\left(T + \sqrt{T^2 - 4d}\right) \qquad \sigma_{qq} = \frac{2d}{T + \sqrt{T^2 - 4d}}. \tag{5.7}$$

If $\det \mathbf{R} > 0$, then $T > 2d + \frac{1}{2}$ and $\sigma_{qq} < \frac{1}{2}$, which means that the quantum state is indeed *squeezed*, since σ_{qq} is less than the variance of the quadrature component in the ground state.

To find the asymptotics of the photon distribution function, we need a simplified version of equation (4.9) for $R_{11} = R_{22}^*$ and $y_1 = y_2^*$

$$\begin{aligned} H_{mm}^{\{\mathbf{R}\}}(y_1, y_2) \approx{}& \left(\frac{mR}{e}\right)^m \exp\left[(1 + 2\zeta^2)(\operatorname{Re}\chi)^2 - \frac{1}{4}(2 + 3\zeta + 4\zeta^2)|\chi|^2\right] \\ & \times \left\{(1+\zeta)^{m+\frac{1}{2}} \cosh\left[2\sqrt{m(1+\zeta)}\operatorname{Im}\chi\right] \exp\left[-\frac{\zeta}{2}(\operatorname{Re}\chi)^2 + \frac{\zeta}{4}(1+\zeta)|\chi|^2\right]\right. \\ & + (1-\zeta)^{m+\frac{1}{2}} \cos\left[2\sqrt{m(1-\zeta)}\operatorname{Re}\chi - m\pi\right] \\ & \left.\times \exp\left[\frac{\zeta}{2}(\operatorname{Re}\chi)^2 - \frac{\zeta}{4}(1+\zeta)|\chi|^2\right]\right\} \end{aligned} \tag{5.8}$$

where $\chi = \sqrt{R_{11}}y_1$. If $\sigma_{pq} = 0$ and $\sigma_{pp} > \sigma_{qq}$, then, due to equations (5.2)–(5.4), we obtain the following expressions for the parameters entering the right-hand side of equation (5.8):

$$R = \frac{2\sqrt{T^2 - 4d}}{2T + 4d + 1} \qquad \zeta = \frac{4d - 1}{2\sqrt{T^2 - 4d}} \tag{5.9}$$

$$\chi = \frac{2\left(T^2 - 4d\right)^{1/4}}{(2T + 4d + 1)^{1/2}}\left[\frac{\langle q\rangle}{\sqrt{T^2 - 4d} - T + 1} + \mathrm{i}\frac{\langle p\rangle}{\sqrt{T^2 - 4d} + T - 1}\right]. \tag{5.10}$$

The oscillations of the photon distribution function manifest themselves to the greatest extent when $\operatorname{Im}\chi = 0$ (i.e. when $\langle p\rangle = 0$ for our choice of the coordinate axes) and $\zeta \ll 1$ and they quickly disappear when $\operatorname{Im}\chi$ becomes large or $\zeta \to 1$. For this reason, we confine ourselves to the case of highly squeezed ($T \gg 1$) and slightly mixed ($\delta \equiv 4d - 1 \ll 1$)

states. Then, the leading terms of the corresponding expansions of equations (5.9) and (5.10) are as follows

$$R \approx 1 - \frac{1}{T} \qquad \zeta \approx \frac{\delta}{2T} \qquad \chi \approx \sqrt{2}\left(\langle q\rangle + \mathrm{i}\frac{\langle p\rangle}{2T}\right).$$

For $\langle p\rangle = 0$, equations (5.1), (5.5) and (5.8) result in the following asymptotical formula (Stirling's formula for $m!$ was used):

$$\mathcal{P}_m \approx \frac{1}{\sqrt{\pi m T}}\left(1-\frac{1}{T}\right)^m \left\{\left(1+\frac{\delta}{2T}\right)^{m+\frac{1}{2}} + (-1)^m\left(1-\frac{\delta}{2T}\right)^{m+\frac{1}{2}} \cos\left(2\sqrt{2m}\langle q\rangle\right)\right\}. \tag{5.11}$$

It is correct provided that the higher-orders terms in the expansions of functions R and ζ in the power series of the small parameter T^{-1} can be neglected. This condition implies, in particular, the relation $m/T^2 \ll 1$. In such a case, equation (5.11) can be written in the following equivalent form:

$$\mathcal{P}_m \approx \frac{1}{\sqrt{\pi m T}}\exp\left(-\frac{m}{T}\right)\left\{\exp\left(\frac{m\delta}{2T}\right) + (-1)^m \exp\left(-\frac{m\delta}{2T}\right)\cos\left(2\sqrt{2m}\langle q\rangle\right)\right\}. \tag{5.12}$$

If $\langle q\rangle = 0$, we have large maximums for the even values of m and small minimums for the odd values of m. The ratio of the neighbouring minimal and maximal values of $\mathcal{P}_m$ equals

$$\mathcal{P}_{\min}/\mathcal{P}_{\max} = \tanh\left(\frac{m\delta}{2T}\right).$$

Consequently, the oscillations disappear when $m\delta/T > 1$ (then $\mathcal{P}_{\min}/\mathcal{P}_{\max} > \frac{1}{2}$). For $\langle q\rangle \neq 0$, the picture of oscillations is more complicated. The oscillations disappear even for $\delta = 0$, if $2\sqrt{2m}\langle q\rangle = (\pi/2) \times$ (odd integer). Moreover, if $2\sqrt{2m}\langle q\rangle = \pi \times$ (odd integer), then maximums take place for odd values of m, whereas minimums take place for even values. (One should remember that the asymptotical formulae with $y_1 \sim \langle q\rangle \neq 0$ derived in the preceding section are valid provided $\langle q\rangle \ll \sqrt{m}$. Therefore, the shift $m \to m+1$ does not change the value of $\cos(2\sqrt{2m}\langle q\rangle)$.)

Parameter $\delta = 4d - 1$ characterizes 'the degree of quantum mixing' due to the relation $\mathrm{Tr}\,\hat\rho^2 = (1+\delta)^{-1/2}$, $\hat\rho$ being the normalized statistical operator of the system. Evidently, the squeezed mixed state is not an equilibrium state. Nonetheless, some conventional 'effective temperature' can be introduced in a more or less unambiguous way if one takes into account that a mixed squeezed state $\hat\rho$ can be obtained from a certain 'initial' equilibrium state $\hat\rho_\beta$ by means of a linear canonical transformation of the 'initial' coordinates q_0 and p_0 [8]

$$\begin{pmatrix}\hat q\\ \hat p\end{pmatrix} = \Lambda\begin{pmatrix}\hat q_0\\ \hat p_0\end{pmatrix} + \begin{pmatrix}\gamma_q\\ \gamma_p\end{pmatrix} \equiv \hat{\mathbf{S}}\begin{pmatrix}\hat q_0\\ \hat p_0\end{pmatrix}\hat{\mathbf{S}}^\dagger \qquad \hat\rho = \hat{\mathbf{S}}\hat\rho_\beta\hat{\mathbf{S}}^\dagger$$

where $\hat{\mathbf{S}}$ is a unitary 'squeezing operator' and Λ is the corresponding *symplectic* matrix. The 'initial' and 'final' variance matrices are related as follows: $\boldsymbol{\sigma} = \Lambda\boldsymbol{\sigma}_\beta\Lambda^{\mathrm{T}}$, Λ^{T} being the transposed matrix. Since any symplectic matrix satisfies the identity $\det(\Lambda\Lambda^{\mathrm{T}}) \equiv 1$, we have

$$d = d_\beta = \frac{1}{4}\left(\coth\frac{\beta}{2}\right)^2 \tag{5.13}$$

β being the dimensionless inverse temperature of the 'initial' equilibrium state. Equation (5.13) just provides an unambiguous definition of the 'effective temperature' for any squeezed state. For 'slightly mixed' states

$$\delta = \left(\sinh\frac{\beta}{2}\right)^{-2} \approx 4\mathrm{e}^{-\beta}$$

so the oscillations of the photon distribution function disappear when $m > T\mathrm{e}^{\beta}$ (provided $m \gg 1$, $T \gg 1$, $\beta \gg 1$).

6. Discussion

Let us outline briefly the problems which still need to be solved. In principle, it is clear how to obtain the generalization of the formulae for the two-dimensional polynomials to the N-dimensional case on the basis of the multidimensional generalization of the Feldheim integral representation. The work in this direction is in progress and the results will be reported elsewhere. However, this approach can be applied for the restricted class of polynomials generated by matrices with positive determinants only. It would be interesting to find the method applicable to polynomials with arbitrary generating matrices (as was performed in [14] for the two-dimensional polynomials of zero arguments). Besides, the asymptotic behaviour of the two-dimensional polynomials for large values of the argument $|y| > \sqrt{m}$ is still unknown (except for the trivial case when $|y| \gg m$).

Appendix

Since the steepest-descent method is usually applied to the integrals of a single variable, let us briefly discuss its applicability to the integral of two variables

$$I = \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \, \exp[F(t_1, t_2)]. \tag{A.1}$$

Assuming t_2 to be a parameter, we may evaluate first the integral over t_1. It equals

$$I_1 = \int_0^\infty \mathrm{d}t_2 \left[\frac{2\pi}{-F_{11}(\tau_1(t_2), t_2)}\right]^{1/2} \exp[F(\tau_1(t_2), t_2)] \tag{A.2}$$

where $\tau_1(t_2)$ is determined from the equation $\partial F/\partial t_1 = 0$ and $F_{ij} \equiv \partial^2 F/\partial t_i \partial t_j$. Designating $\Phi(t_2) = F(\tau_1(t_2), t_2)$, we find the saddle point for the integral (A.2) from the equation $\partial\Phi/\partial t_2 = 0$. However, this equation is equivalent to $\partial F/\partial t_2 = 0$ due to the relation

$$\frac{\partial \Phi}{\partial t_2} = \frac{\partial F}{\partial t_2} + \frac{\partial F}{\partial t_1}\frac{\mathrm{d}\tau_1}{\mathrm{d}t_2}.$$

Consequently, the saddle point for the two-dimensional integral can be determined directly from the system of equations

$$\frac{\partial F}{\partial t_1} = 0 \qquad \frac{\partial F}{\partial t_2} = 0. \tag{A.3}$$

As to the pre-exponential factor, it equals $2\pi[F_{11}(\tau_1(\tau_2),\tau_2)\Phi''(\tau_2)]^{-1/2}$ where τ_1 and τ_2 are the solutions to equation (A.3). Due to the definition of function $\Phi(t_2)$, we get

$$\Phi''(\tau_2) = F_{22} + 2F_{12}\frac{\mathrm{d}\tau_1}{\mathrm{d}t_2} + F_{11}\left(\frac{\mathrm{d}\tau_1}{\mathrm{d}t_2}\right)^2 + \frac{\partial F}{\partial t_1}\frac{\mathrm{d}^2\tau_1}{\mathrm{d}t_2^2}.$$

The last term in this expression becomes zero at the saddle point. To find the derivative $\mathrm{d}\tau_1/\mathrm{d}t_2$, we take into account that the equation $\partial F(\tau_1(t_2),t_2)/\partial t_1 = 0$ must be satisfied identically for all values of t_2. Differentiating this identity with respect to t_2, we get $\mathrm{d}\tau_1/\mathrm{d}t_2 = -F_{12}/F_{11}$. Consequently, $\Phi''(\tau_1,\tau_2) = F_{22} - F_{12}^2/F_{11}$. Therefore, the asymptotics of the integral (A.1) reads

$$I \approx 2\pi[F_{11}F_{22} - F_{12}^2]^{-1/2}\exp[F(\tau_1,\tau_2)]. \tag{A.4}$$

This expression shows that the most direct way to obtain the asymptotics of the integral (A.1) is as follows. First, one should find the saddle point in the two-dimensional space from the system of equations (A.3). Then, one should develop function $F(t_1,t_2)$ in Taylor's series in the vicinity of the saddle point up to second-order terms and calculate the arising Gaussian integral. The result will be given just by equation (A.4). Note that both function F and the coordinates of the saddle point (τ_1,τ_2) may be complex.

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